

*Life of Fred*<sup>®</sup>  
*Calculus*  
*Expanded Edition*

Stanley F. Schmidt, Ph.D.



Polka Dot Publishing

## *What Calculus Is About*

I remember standing in the college bookstore at the beginning of my freshman year. I pulled a beginning calculus textbook off the shelf and opened it. What a frightening sight it was.

The pages were filled with strange symbolism like  $\int_{x=0}^2 (4 - x^2) dx$  and  $\frac{\partial \phi}{\partial x} = M$ . It might as well have been in Turkish. No one else in my family had ever studied calculus, so there was no one to give me an overview of what lay ahead. All I was told was that anyone who wanted to study any of the topics that I was even remotely interested in would want to have a grounding in calculus. Even business majors going on for a master's degree were required to study it.

But that didn't tell me what it was. I looked at some of the problems in that old textbook:

11. Find  $dy/dx$  for  $y = \sin x$ .
18. Determine the eccentricity of  $(y + 5)^2 + 4(x - 5)^2 = 1$ .
22. Solve  $y' = \tanh x$

From my trig class I recognized "sin x" and knew that it didn't have theological overtones in this context. But I was still at a loss as to what calculus was about or why I needed to learn it. Was this stuff useful? Would I find a need for it in my everyday life?

Yes. The book you now hold in your hands shows that every aspect of calculus can arise in the course of daily living. If you've ever fallen into a vat of cheese soup (Chapter 19) or tried to run a thousand pounds of ammo through a customs station (Chapter 9) you know what I mean.

So what's calculus? In a sentence:

If it moves at a varying speed,  
if it has a curvy shape,  
if it has a maximum that you'd like to find, or  
if it involves adding up an infinite number of terms,  
then you're probably looking at calculus.

## *A Note to Students*

**H**i! This is going to be fun.

When I studied calculus my teacher told the class that we could reasonably expect to spend thirty minutes per page to master the material in the old calculus book we used. With the book you are holding in your hands, you will need two reading speeds: thirty minutes per page when you're learning calculus and whatever speed feels good when you're enjoying the life adventures of Fred.

This book has five parts to it:

1. The life adventures of Fred
2. *Your Turn to Play*
3. *Further Aids*
4. Answers
5. Index

Start on the first page of the life adventures of Fred and things will explain themselves nicely.

After 24 chapters you will have mastered all of lower-division (freshman and sophomore) calculus.

## *A Note to Teachers*

This book wasn't written with you in mind. Instead it was created for those who will be learning calculus from it. There are a thousand banal, dignified calculus books with their overwhelming sense of restraint and propriety that present the material as it has always been presented:

definition  
theorem  
proof  
corollary  
definition  
lemma  
theorem  
proof  
definition

.  
.  
.

with such a lack of excitement that even a rock would be bored.

There is no reason why the study of calculus should have suffering as its dominant motif—as if our students were children in some Dickens novel. Notwithstanding any objections of the American Dental Association, a spoonful of sugar really doesn't do any harm. In the years I taught calculus, I often looked out at the faces of the students and thought to myself that these individuals are not just *Homo habilis* (the toolmaker, the worker) but *Homo ludens* (the playful).

This text will also make your life more pleasant. You haven't been forgotten. Here are some secrets between you and me:

1. *Cities* will be your real friend. Six cities are found at the end of each chapter in the main text. Each city is a set of problems and questions which may take your students 20–30 minutes to work through. You can assign one or more of the cities as homework. In this Expanded Edition all of the answers to the problems in *Cities* are supplied. They will have immediate feedback as to whether they've mastered the material.

2. The main meat of calculus is found in the first section of the book which tells of the life adventures of Fred and in the *Your Turn to Play* which contains lots of completely-worked-out problems. Many instructors assign the problems from the *Your Turn to Play* section. These provide the graded exercises that many beginning students require before tackling the more substantial problems in *Cities*. (*Your Turn to Play* begins at page A-401.)
3. You can make this course as high-powered as you wish by including whatever material you like from the *Further Aids* section. (*Further Aids* begins at page B-441.)
4. Every calculus book leaves out some topics. Very few are left out of *Life of Fred*. (Check the index for your favorite topics.) Furthermore, there are many ideas/topics/themes/presentations that can't be found anywhere else. The rigor is also here with much of it in the *Further Aids* section of the book so that *you* may decide how much of it you'll include in the course you're teaching.
5. On these pages you and your students will find all of calculus folded into the story of the early years of Fred. The topics of calculus arise naturally out of his life experiences.
  - When Fred and his doll Kingie are playing on the kitchen floor (while his mother makes cookies) he invents a game to play with Kingie. That game evolves into the Mean Value Theorem.
  - Visiting the buffet at a casino in Oz, Fred and his friends encounter hyperbolic trig functions wherever they turn.
  - Line integrals arise naturally when Fred does some investigative work at the law library in his attempt to get expelled from kindergarten.

You might enjoy this book as much as your students do!

# Contents

(Topics in *small italics* are covered in the *Further Addo* section of the book at the page numbers indicated.)

Chapter 1	Functions. . . . .	15
	Domain, Codomain, Range, Onto	
Chapter 2	Limits. . . . .	27
	$\epsilon$ - $\delta$ definition 443	
Chapter 3	Speed. . . . .	37
	Average Speed vs. Instantaneous Speed	
Chapter 4	Slope. . . . .	45
	Tangent Lines	
Chapter 5	Derivatives. . . . .	55
	Maximums/Minimums	
	Product/Quotient/Chain Rules	
	<i>proofs of the product/quotient/chain rules 448</i>	
Chapter 6	Concavity. . . . .	73
	Second Derivatives	
	Asymptotes	
Chapter 7	Trig. . . . .	89
	Tests for Extrema	
Chapter 8	Related Rates. . . . .	103
	Implicit Differentiation	
	Explicit/Implicit/Parametric	
Chapter 9	Curvature. . . . .	117
	Mean Value Theorem	
	<i>proof of the MVT 456</i>	
	<i>L'Hospital rule 458</i>	
	Acceleration	
	Antiderivatives	

Chapter 10	Integrals. . . . .	137
	Fundamental Theorem of Calculus	
	<i>proof of FTC 459</i>	
Chapter 11	Area. . . . .	151
	Parametric Forms for Area and Length	
	Improper Integrals	
Chapter 12	Work. . . . .	169
	Solids of Revolution	
	Torque	
Chapter 13	Centroids. . . . .	183
	Differentials	
	Average Value of a Function	
	Integration by Parts	
	Moments of Inertia	
Chapter 14	Logs. . . . .	199
	Probability Density Functions	
	<i>definition of e 464</i>	
	<i>bounded increasing sequences 466</i>	
Chapter 15	Conics. . . . .	211
	Hydrostatic Force	
	<i>oblique asymptotes 470</i>	
Chapter 16	Infinite Series. . . . .	225
	Tests for Convergence	
Chapter 17	Solids of Revolution. . . . .	246
	Trig Substitutions	
	Surface Area	
	Arc Length	
Chapter 18	Polar Coordinates. . . . .	263
	Alternating Series	
	Power Series	
	Evaluating Integrals Using Substitutions	
	<i>partial fractions 483</i>	
	Maclaurin and Taylor Series	
	<i>remainder formula for Taylor 485</i>	

Chapter 19	Hyperbolic Trig. . . . .	286
	Separating the Variables in Differential Equations	
	Numerical Integration	
Chapter 20	Vectors. . . . .	305
Chapter 21	Partial Derivatives. . . . .	319
	Chain Rule with Intermediate Variables	
	<i>Lagrange multipliers 489</i>	
Chapter 22	Double Integrals. . . . .	337
	Cylindrical Coordinate System	
	<i>spherical coordinates 491</i>	
Chapter 23	Vector Calculus. . . . .	357
	Gradient	
	Directional Derivative	
	Line Integrals	
	Green's Theorem	
	<i>flux of a vector through a surface 497</i>	
	<i>Divergence theorem 505</i>	
	<i>Stokes's theorem 510</i>	
Chapter 24	Differential Equations. . . . .	381
	Variables Separable	
	<i>exact and integrating factors 512</i>	
	Orthogonal Trajectories	
	First Order Linear	
	<i>Bernoulli's equation 440</i>	
	Second Order	
Your Turn to Play. . . . .		A-401
Further Ado. . . . .		B-441
Answers. . . . .		C-519
Index. . . . .		582



# Chapter One

## Functions

Once upon a time, a long time ago, on the western slopes of the Siberian Mountains, there lived Fred's parents. However, they weren't called Fred's parents since Fred hadn't been born yet.

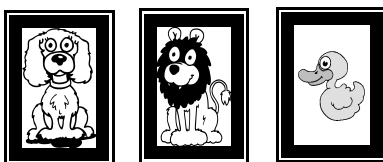
But one day, more recently than a long time ago, the stork delivered Fred. The lucky couple, Mr. & Mrs. Gauss, discovered that they were his parents.




At least Mrs. Gauss (rhymes with "house") thought she was a parent. Staring at Fred, she chattered, "Oh, isn't our baby beautiful!"

Mr. Gauss frowned and said, "He doesn't look a bit like me." Mrs. Gauss didn't get the drift of what her husband was saying. She responded, "Of course he doesn't. He's just a little baby, all red and wrinkly, and, besides, he was very young when he was born." Fred's father rolled back his eyes, turned, and left the room.


Mrs. Gauss carried him around a while and then, not knowing what to do with him, put him back in his crib. She had high hopes that her little tyke would grow up to be a country western singer. After she tucked him in, she handed him a new toy. It was a box with three buttons on it. Each button had an animal printed on it.



When Fred hit the button with the dog on it, the box sounded, "Bow-wow!" When he tried the lion, "Roar!" The duck, "Quack!" This was Fred's first encounter with the idea of function. He found out that EVERY time he touched the  he heard, "Quack!"

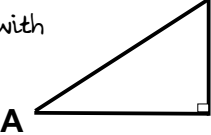
Here is how he summarized in his head what he knew about his animal-toy function:

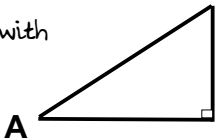
1. There are two sets involved: the set of animal buttons {dog, lion, duck} and the set of animal sounds {Bow-wow, Roar, Quack}.
2. Every time I hit the lion I get a sound and it's always the same sound.

Fred was fascinated by this idea of function. You start with two sets and for each element of the first set there is exactly one element of the second set which corresponded to it. Fred looked around his study (crib) and invented a new function. His first set contained the things in his crib and the second set was colors. He saw his sheet and that matched up with “white” in the second set. His  matched up with “yellow.” The bars

on his crib also matched up with “yellow.”

Can two different things be “yellow”? Yes. The only critical thing for the idea of a function is that each element in the first set have exactly one **image** in the second set. It's okay if two different elements in the first set have the same image.

Fred thought to himself This is baby stuff! I'm one day old and I should be able to think of a more sophisticated example of a function. He thought of his diaper which was in the shape of a right triangle. He labeled one of the acute angles with the letter A and created the following function: For any acute angle A, draw a diaper—I mean a triangle—with  one of the acute angles being A.



Then Fred continued measure the length of the side opposite and divide that length by the length of the hypotenuse. When Fred set angle A equal to  $35^\circ$ , the result of using his function (namely, drawing a triangle with a  $35^\circ$  angle and dividing the opposite side by the hypotenuse) gave him a result of 0.5735764. Fred was very good at measuring lengths. He called

this function that he invented the sine function and he wrote  $\sin(35^\circ) = 0.5735764$ .

But what if he had used a bigger triangle? Would the answer come out differently? No. He knew he'd get the same answer every time since any two right triangles with  $35^\circ$  angles would be similar (something he had read in his geometry book) and similar triangles are triangles in which the sides are proportional.

Now since every element in the first set, which is the set of all acute angles, has a unique image in the second set, Fred knew that he was dealing with a genuine function.

When Mrs. Gauss came in to see how Fred was doing, she found that he had drawn triangles all over his bed sheets.

As she looked down into Fred's study and made little "goo-goo-goo" sounds at him, he said, "Mom, let's play a little game. We'll call it, Guess the Function."

Fred continued, "I'm thinking of a function which I'm going to call 'f' and I'm going to give you some examples and you try and guess what the function is. Are you ready?"

Mrs. Gauss nodded but wasn't sure what Fred was talking about. Then Fred wrote on a sheet:

$$f(7) = 15$$

$$f(3) = 7$$

$$f(6) = 13$$

$$f(100) = 201$$

Mrs. Gauss looked at what he had written. She looked at it for a long time. Finally she said, "Are you hungry?"

Fred shook his head and gave her some more examples:

$$f(300) = 601 \qquad f(12) = 25$$

$$f(0) = 1 \qquad f(4.5) = 10$$

Finally after waiting several more minutes, he gave her the “hint”:

$$f(x) = 2x+1.$$

He tried to explain to her that a function is any rule which associates to each element of the first set exactly one element of the second set. The object of the game was to guess the rule.

If you would like to play Guess the Function see page A-401 in *Your Turn to Play* in the back of this book. If you are reading this book to learn calculus, and not just enjoy the adventures of Fred, every opportunity for you to take *Your Turn to Play* should be taken. The nice part is that complete solutions are given for all the questions. The *Your Turn to Play* is one of the easiest ways to learn calculus.

His mother left the room humming a tune that didn’t seem to have a melody. Fred was alone to think functions some more. He wanted to invent some more of them.

First, I should think of two sets Fred thought and then I’ll think of a **mapping** (which is another name for function) between them. He cast his eyes around the room and spotted a huge stack of old country western fan magazines that his mom had stacked up in the corner. We’ll let that be the first set. (The first set is called the **domain** of a function.)

From his vantage point in his crib he could see part of the kitchen counter. On it sat a blender, a coffee pot and a mixer. I’ll let these be



the elements of my second set. (The second set is called the **codomain** of a function.)

Since a function is ANY rule which associates to each element of the domain, an element of the codomain, Fred could think of hundreds of possible functions:

**First example:** to every magazine with the letter “R” on the front cover assign those magazines to the coffee pot and all the rest of the magazines assign to the mixer.

**Second example:** All magazines less than 50 pages long assign to the blender; those with 50–75 pages assign to the coffee pot and the rest to the mixer. Then the **image** of a magazine with 58 pages would be the coffee pot.

**Third example:** The oldest magazine (which happened to be the July 1952 issue) is assigned to the blender. The youngest magazine is assigned to the coffee pot and all the rest are assigned to the mixer. In this case every element of the codomain was the image of at least one element of the domain. Such a function is said to be **onto** the codomain. (In Fred’s first example, that function wasn’t onto the codomain since no magazine had the blender as an image.)

**Fourth example:** Assign all the magazines to the mixer. This, of course, is not onto the codomain. The set of all images of a function is called the **range** of the function. The range of this function is just the mixer. If the range equals the codomain, then the function is onto.

That’s a lot of new words and concepts (unless you had them in your algebra classes). Let’s turn to *Your Turn to Play* on page A-402 and play with those ideas until they become a bit more comfortable.

(If you want to read more about functions, see page B-441 in *Further Ado* in the back of this book. If you’ll read the first three paragraphs on that page, it’ll explain what *Further Ado* is all about.)



If you have a yellow highlighter, please color the duck on the previous pages.



It’s traveling time. We will have the opportunity to visit six cities in the next several pages. (If you haven’t looked at *Your Turn to Play*, these questions will be tougher than they need to be.)

Good news! Answers for the *Cities* are supplied right after the six Cities. If your purpose is to learn calculus, then do the problems first and then check your answers.

# Index

- 1-1 correspondence. . . . . A-403
  - between the natural numbers and the rational numbers . . . . . B-442
  - not between the natural numbers and the real numbers. . . . . B-442
  - two sets have the same number of elements in them. . . . . B-441, C-543
- absolute maximum. . . . . 66
- absolutely convergent series . . . . . 268
  - commutative law does hold . . . . . C-563
- acceleration. . . . . 126, A-415
- alternating series. . . . . 267, A-429
  - absolutely convergent series . . . . . A-429
  - conditionally convergent series. . . . . A-429
  - convergence. . . . . 267
  - truncating. . . . . A-429
- angular speed. . . . . 190
  - and related rates. . . . . 191, 211
- antiderivative. . . . . 127
  - guess-and-by-gosh method . . . . . A-427
  - of  $1/x$ . . . . . 202
  - substitutions. . . . . A-427
  - trig substitution. . . . . A-428
- applications of integration. . . . . 255, 256
- arc length. . . . . 254, 255
  - cardioid. . . . . 266
  - in parametric. . . . . 159, 255
  - rainbow. . . . . 161
  - rectangular form. . . . . 159
- Archimedes. . . . . B-506
- area. . . . . A-417
  - ellipse. . . . . 157
  - parametric form. . . . . 157
  - Tinker Creek. . . . . 153
  - under a curve. . . . . 199
- area under a curve. 153-154, 255
  - in parametric. . . . . 255, A-418
- Aristotle. . . . . C-536
- asymptote. . . . . 218, A-408
  - def of horizontal asymptote . . . . . B-453
  - horizontal. . . . . 75
  - oblique. . . . . 218, B-470
  - vertical. . . . . A-408
- average value of a function. . . . . 186, 256, A-420
- BASIC program
  - for approximating e. . . . . B-465
  - for integration. . . . . 160
  - for partial sums. . . . . 231
- Bernoulli's equation. . . . . A-440
- binomial formula. . . . . B-468
- boundary conditions. . . . . 386
- cap. . . . . 73
- cardioid. . . . . 266, A-429
  - arc length. . . . . 266
- catenary. . . . . 289

## *Index*

- center of gravity. . . . . 184, A-419
- center of mass. . . . . 184
- centroid. . . . . 184, 256, A-419
- chain rule. . . . . 64, A-406
  - proof. . . . . B-448
  - with several intermediate variables. . . . . 331-332
- change of variable. . . . . 185, 247
- chart
  - delta process. . . . . 40, 48, 58, 93
  - derivative of  $1/x$ . . . . . 203
  - derivatives. . . . . 64
  - favorite integration
    - substitutions. . . . . 270
  - implicit vs. explicit. . . . . 110
  - limit of sine. . . . . 29
  - Maclaurin series. . . . . 278
  - Taylor series. . . . . 280
  - three tests for minimums. . . . . 90
  - vertical asymptote. . . . . 81
- circle. . . . . 215
- closed curves. . . . . 374
- closed interval. . . . . 124
- codomain. . . . . 18
- Comparison Test. . . . . 233
- concave. . . . . 73
- concavity. . . . . 73, A-408
- conditionally convergent series
  - . . . . . 268
  - commutative law doesn't hold
    - . . . . . C-562
- cone
  - definition. . . . . 218
  - filling. . . . . 56, 57
- conic sections. . . . . 216, A-425
  - in polar. . . . . B-481
  - pointed the "wrong way"
    - . . . . . 221
    - rotated. . . . . 220, A-425
    - translated. . . . . 220
- conservative field. . . . . 371
- constant of integration. . . . . 193
- continuous function. . . . . 33
- contrapositive. . . . . B-474
- convergent. . . . . 229, A-426
- convex. . . . . 73
- cosh  $x$ . . . . . 288, A-431
  - derivative. . . . . 291
- critical points. . . . . 66
- cross product. . . . . B-509
- curl  $F$ . . . . . B-511
- curvature. . . . . 118-119, A-414
  - formula. . . . . 120
  - formula (in parametric form)
    - . . . . . 121
- cycloid. . . . . 110
  - parametric form. . . . . B-455
- cylindrical coordinate system
  - . . . . . 350
- deductive logic. . . . . B-474
- definite integrals. . . . . 193, A-416
  - evaluating. . . . . 143
  - steps to setting them up. . . . . 144
- del. . . . . 361
- delta. . . . . 37
- delta process. . . . . 39, 93
  - for a cubic. . . . . 47
  - for  $\log x$ . . . . . 203
  - for  $x$  to the  $n$ th power. . . . . 58
  - for  $y = 6$ . . . . . B-448
  - for  $y = \sin x$ . . . . . 93
  - Your Turn to Play examples
    - . . . . . A-405
- density
  - in a solid. . . . . 174

## *Index*

- probability density function
  - ..... 204-206
  - variable along a length. . . . 141
- derivative. .... 73
  - arc trig functions..... 92
  - definition..... 57
  - dy/dx..... 57
  - hyperbolic trig functions
    - ..... 291, A-432
  - in parametric form. .... 111
  - partial. .... 322
  - sine by delta process..... 93
  - trig functions.....
    - ..... 91-92, A-409-A410
  - trig functions (applications)
    - ..... A-411-A-412
  - variable in the exponent
    - ..... 206
- differential equations. ....
  - ..... 104, A-432
  - boundary condition..... 385
  - exact. .... B-512
  - first-order linear. .... A-439
  - homogeneous. .... A-437
  - integrating factors . . . B-512,
    - .. B-514-515, B-517-518
  - mixing problem..... 389
  - orthogonal trajectory..... 387
  - second order. .... 393
  - separating the variables.....
    - ..... 292, 382
  - variables separable. . . . A-437
  - what lies ahead. .... B-518
- differential form..... 185, B-462
- direction of maximum change
  - ..... 361
- directional derivative. .... 364
- distributive
  - finite case. .... 233
  - formulas for vectors. .... 314
  - infinite case..... 233, B-477
- div F..... B-506
- divergence. .... B-506
- divergence theorem. .... B-505
- divergent. .... 229
- diving board (off the)... 126-127
- domain of a function. .... 28
- dot product of vectors. .... 310
  - formulas. .... 314
- double integral. .... 341, A-434
  - finding area..... 345
  - finding weight of area with
    - variable density. .... 346
    - moment of inertia. .... 347
    - torque. .... 347
- Dr. Johnson. .... 257
- ds (arc length)..... 159
  - in polar. .... 264
- dS (surface). .... B-499
  - in polar. .... B-503
- dummy variable. .... 295
- $d^2y/dx^2$ . .... 73
- e. .... 203
  - definition..... B-464-B-465
- eccentricity
  - in polar. .... B-482
  - in rectangular. .... B-472
- ellipse..... 216
  - eccentricity. .... B-471
  - foci. .... B-471
  - in polar. .... B-481
  - parametric form. .... 157
- energy of rotation..... 190



## *Index*

- epsilon-delta. . . . . B-444-B-445
  - arguments using. . . . . B-446,  
B-447, B-450, B-451,  
B-467, B-470, C-520,  
C-529, C-535
- equation of a plane through a  
point and normal to a  
vector. . . . . 362
- exact differential equation  
. . . . . B-512
  - potential function. . . . . B-512
- explicit relation. . . . . 108, A-413
- exponential functions. . . . . A-421
- extrema. . . . . 60
- first derivative test. . . . . 90
- first moment. . . . . 178
- first-order linear. . . . . 390, A-439
- flux of a vector through a surface  
. . . . . B-497
- focus. . . . . 216
- folium of Descartes. . . . . 108
- function of two variables. . . . . 321
- functions. . . . . 16, B-441
  - 1-1. . . . . A-403
    - codomain. . . . . 18
    - continuous. . . . . A-404
    - domain. . . . . 18, A-402
    - image under a function. . . . . 19
  - increasing bounded functions  
. . . . . B-466
  - inverses. . . . . A-403
  - onto. . . . . 19
  - ordered pairs definition  
. . . . . A-403
  - range. . . . . 19
- Fundamental Theorem of  
Calculus. . . . . 144, 252
  - needs continuous function  
. . . . . 162
  - proof. . . . . B-459-461
- Gauss, Karl Friedrich. . . . . B-507
- Gauss's theorem. . . . . B-505
- general operating rule for doing  
integration in polar form  
. . . . . 266
- geometric series. . . . . 228
  - sum. . . . . 229, B-473
  - when convergent. . . . . 230
- Gödel. . . . . 228
- golden mean. . . . . C-536
- Goldilocks. . . . . C-536
- grad  $u$ . . . . . 360
- gradient. . . . .
  - . . . . . 360, A-435-A-436, B-492
  - length of the vector. . . . . 363
  - normal to a surface. . . . . B-497
- gravity (motion under). . . . .
  - . . . . . 126-127
- greatest-integer-in function. . . . . 44
- Greek alphabet. . . . . 295
- Green, George. . . . . 373
- Green's theorem. . . . . 373
- Green's theorem in space  
. . . . . B-505
- guess a function. . . . . 18, A-401
- guess-and-by-gosh method. . . . .
  - . . . . . 248, A-427
- homogeneous equations. . . . . A-437
- horizontal asymptote. . . . . 75, B-453
- hydrostatic force. . . . .
  - . . . . . 213, 256, A-424

## Index

- hyperbola. . . . . 217
  - asymptotes. . . . . 218
  - eccentricity. . . . . B-471
  - foci. . . . . B-471
  - in polar. . . . . B-481
- hyperbolic trig functions. . . . . 288
  - derivatives. . . . . 291
  - formulas. . . . . 288
- Ice Cream Cone problem (surface integrals). B-501, B-502, B-504, B-505, B-507
- image. . . . . 16
- implications (if-then). . . . . B-474
  - contrapositive. . . . . B-474
  - converse. . . . . B-474
  - inverse. . . . . 231, B-474
- implicit differentiation. . . . . 108
- implicit relation. . . . . 108, A-413
  - finding tangent. . . . . 108
- improper integral. . . . . A-418
- incomplete elliptic integral of the second kind. . . . . 161
- indefinite integral. . . . . 193
- inductive reasoning. . . . .
  - . . . . . B-475-B-476
- infinite series. . . . . A-426
  - “last term”. . . . . 230
  - absolutely convergent series
    - . . . . . 268
  - conditionally convergent series. . . . . 268
  - convergent. . . . . 229
  - distributive law. . . . . 233
  - divergent. . . . . 229
  - harmonic series. . . . . 233
  - integral test. . . . . 234
  - Limit Comparison Test. . . . .
    - . . . . . 233, B-476
  - Maclaurin series. . . . . A-431
  - partial sums. . . . . 231
  - Ratio Test. . . . . 239
  - rules for convergence. . . . .
    - . . . . . 230, 233-234, 239
  - Taylor series. . . . . 281, A-431
- initial condition. . . . . 383
- instantaneous speed. . . . . 39
- integers. . . . . B-469
- Integral Test. . . . . 234
- integrals
  - change of variables. . . . . 247
  - definite. . . . . 193, A-416
  - favorite integration
    - substitutions. . . . . 270
  - improper. . . . . A-418
  - in polar form (general operating rule). . . . . 266
  - indefinite. . . . . 193
  - limits. . . . . A-416
  - line. . . . . 367, A-436-A-437
  - power series. . . . . A-430
  - substitutions. . . . . A-430
  - table of all the applications
    - . . . . . 255, 256
  - trig substitution. . . . .
    - . . . . . 248-250, A-428
- integrand. . . . . 161
- integrating factors. . . . . B-514, B-515, B-517, B-518
  - little black book. . . . . B-517
- integration
  - changing the order. . . . . B-504
  - over a surface. . . . . B-497
- integration by parts. . . . . 188, A-421
  - proof. . . . . 189
  - used twice. . . . . 190
- intermediate variables. . . . . 331

## Index

- interval of convergence. . . . . 274  
    differentiating or integrating  
        within. . . . . 275
- isothermally. . . . . 362
- iterated integral. . . . . 341
- kinetic energy. . . . . 190
- Lagrange multipliers. . . . .  
    . . . . . 328, B-489  
     $\lambda$ . . . . . B-490
- Law of Cooling. . . . . 385
- least upper bound. . . . . B-467
- Leibnitz. . . . . 229
- lemniscate. . . . . 264  
    in polar. . . . . 264
- length of a vector. . . . . 309
- length of the gradient vector  
    . . . . . 363
- L'Hospital's rule. . . . . B-458
- $\lim (\sin\theta)/\theta$ . . . . . 93
- limaçon. . . . . A-429
- limit. . . . . 32  
    definition (epsilon–delta)  
        . . . . . B-443  
    of a function. . . . . 28  
    of a product. . . . . B-451  
    of a sum. . . . . B-450  
    one-sided. . . . . 162
- Limit Comparison Test. . . . . 233  
    proof. . . . . B-476
- line integral. . . . . 367, A-436, B-492
- $\ln x$ . . . . . 203
- local maximum. . . . . 66
- logarithmic differentiation  
    . . . . . A-422
- logistics curve. . . . . 388
- logs. . . . . 203, A-421  
    definition. . . . . B-463
- long-time stories. . . . . 235-237, 239
- M (torque). . . . . 178
- Maclaurin series. . . . . 278, A-431
- mappings. . . . . 18
- maximums. . . . . 60  
    absolute. . . . . 66  
    along curves in a three-  
        dimensional surface  
        . . . . . 328  
    angle of truck sign. . . . . 90  
    dog's play area. . . . . 140  
    height. . . . . 126  
    hemp plant yield. . . . . 77  
    local. . . . . 66, 321  
    on an interval. . . . . A-407  
    tests for surfaces in three  
        dimensions. . . . . 323, 327
- Mean Value Theorem. . . . .  
    . . . . . 124, A-415  
    proof. . . . . B-456  
    Rolle's theorem (lemma)  
        . . . . . B-456  
    used for approximation. . . . . 128
- minimums. . . . . 60  
    bubbles lost. . . . . 89  
    fetching the beer. . . . . 62  
    paper used. . . . . 78  
    shortest route across the  
        tundra. . . . . 97  
    sound reaching Fred's ears  
        . . . . . 125
- mixed partial derivative. . . . . 324
- moment of inertia. . . . .  
    . . . . . 189, 256, A-421  
    of a banana. . . . . 190  
    using double integrals. . . . . 347
- MVT. . . . . 124, A-415  
    proof. . . . . B-456  
    used for approximation. . . . . 128

## *Index*

- $M_y = N_x$ ..... 371  
     exact differential equation . . .  
         ..... B-512  
 Napier, John..... C-554, C-561  
 natural logs..... 203  
 Newton..... 229, 385, B-506  
 normal to a surface..... B-497  
 numerical integration.....  
     ..... 294, A-432  
     memory aids. .... 299  
     Simpson's rule. .... 298  
     trapezoidal rule..... 297  
 one-sided limit. .... 162  
 one-to-one. .... A-403  
 open interval..... 124  
 p-series..... 235  
 Pappus. .... 250  
 Pappus' theorem..... 250  
 parabola. .... 217  
     in polar. .... B-481  
 parametric representation... 110,  
     A-413  
 partial derivative. .... 322  
 partial fractions. .... 271  
     a brush-up. .... B-483, B-484  
 path-independent. .... 376  
 point of inflection... 74, A-408  
 point-slope equation of the line  
     ..... 49  
 polar coordinates. ... 264, A-429  
     a brush-up. .... B-478, B-480  
     double integration... 348-349  
     finding volume. .... 348-349  
     formulas. .... B-480-482  
 position vector. .... B-494  
 potential function... 372, B-492  
 power series. .... 272, A-430  
     interval of convergence... 274  
     Maclaurin. .... 278  
     Taylor. .... 281  
 probability density function  
     ..... 204, 256, A-421  
 product rule. .... 64, A-406  
     proof. .... B-448  
 Product Rule song. .... 74  
 projection of a vector..... 310  
     length. .... 311  
 pure second partial derivative  
     ..... 324  
 quotient rule . .... 64, A-406  
     proof. .... B-448  
 rabbit and the wall. .... B-466  
 radian measure..... 92  
     definition..... B-453  
     dimensionless. .... 191  
 radius of convergence. .... 274  
 range..... 19  
 rational numbers..... B-469  
 rationalizing the numerator. . 292  
 rectangular rule. .... 296  
 related rates. .... 104, A-412,  
     A-413, A-417  
     and angular velocity. ....  
         ..... 191, 211  
     distance to the Christmas tree  
         ..... 106  
     of the gas cloud..... 105  
     of the squab..... 103  
     silo. .... 154  
     surface area..... 156  
     Toddy's body. .... 155  
 relative maximum. .... 65  
     for surfaces in three  
         dimensions..... 327  
 Rolle's theorem. .... B-456  
 Root Test. .... 239

## Index

- saddle point. . . . . 324-325
- scalar. . . . . 306
- scalar field. . . . . 366
- scalar function. . . . . 360
- scalar multiplication. . . . . 306
- Schröder-Bernstein theorem . . . . .  
. . . . . B-443
- secant line. . . . . 46
- second derivative test. . . . . 73, 90
- second moment. . . . .  
. . . . . 189, 256, A-421
- second order differential  
equations  
    lacking the dependent variable  
        . . . . . 393  
    lacking the independent  
        variable. . . . . 395
- separating the variables. . . . .  
. . . . . 292, 382
- simple paths. . . . . 374
- Simpson's rule. . . . . 298
- $\sinh x$ . . . . . 288, A-431  
    derivative. . . . . 291
- slope of a line. . . . . B-447
- slope of tangent line. . . . . 46-47
- slopes of tangents to curves  
. . . . . A-405
- smooth paths. . . . . 374
- snake (weight). . . . . 141, 143
- solid of revolution. . . . . 246
- speed. . . . . 38  
    instantaneous. . . . . 39
- spherical coordinates. . . . . B-491  
    volume. . . . . B-492
- Stokes's theorem. . . . . B-510-511
- surface area. . . . . A-428  
    in polar. . . . . 264  
    of a paint can. . . . . 252-253  
    of revolution. . . . . 251, 256
- surface integral. . . . . B-497, B-498,  
    B-500, B-502-504
- tangent to a curve. . . . . 45  
    curve in implicit form. . . . . 108
- Taylor series. . . . . 281, A-431  
    remainder formula. . . . . B-485
- Tinker Creek. . . . . 153
- torque. . . . . 177, A-418  
    of a bar. . . . . 255  
    of a pb&j sandwich. . . . .  
        . . . . . 187, 189
- tractrix curve. . . . . 292
- transcendental numbers. . . . . B-469
- trapezoidal rule. . . . . 297
- trig substitutions. . . . . 248-250
- triple integrals. . . . . A-435
- unit tangent vector. . . . . B-494
- unit vector. . . . . 311  
    *i*, *j*, and *k*. . . . . 313
- variable density. . . . . 141
- variables separable. . . . . A-437
- vector. . . . . 306  
    an algebraic view. . . . . B-486  
    components. . . . . B-488  
    dot product. . . . . 310  
    formulas. . . . . 313-314  
    length. . . . . 309  
    position vector. . . . . B-494  
    projection. . . . . 310, 312  
    scalar product. . . . . 310  
    subtraction. . . . . A-432  
    unit. . . . . 311  
    vector product. . . . . B-509  
    zero vector. . . . . 313, A-433
- vector addition. . . . . 307  
    an algebraic view. . . . . B-487  
    associative. . . . . 308  
    formulas. . . . . 313-314

## *Index*

- vector field. . . . . 366
- velocity
  - hitting the ground. . . . . 126
  - limiting. . . . . 289
- velocity vector. . . . . B-494
- vertical asymptote. . . . . 81
- vertically simple. . . . . C-560
- volume
  - cylindrical shells method
    - . . . . . 247
    - of revolution. . . 173, 176, 255
    - polar coordinates. . . . 348-349
    - solid of revolution. . . . .
      - . . . . . 246, 256, A-418
    - solid with constant height
      - . . . . . 339
    - solid with variable height
      - . . . . . 340, 344
    - spherical coordinates. . B-492
  - water pressure. . 213, 256, A-424
  - weight of area with variable
    - density. . . . . 346
  - weight of length with variable
    - density. . . . . 255
  - wire skating. . . . . 253
  - work. . . . . 255, A-418
    - along a curved path. . . . . 366
    - compressing a spring. . . . 171
    - defined. . . . . 169
    - lifting Toddy to Oz. . . . .
      - . . . . . 169, 171
    - pumping water out. . 173, 255
  - x-y-z axes. . . . . 320
  - y". . . . . 73